

Full characterization of the fractional Poisson process

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Abstract –The fractional Poisson process (FPP) is a counting process with independent and identically distributed inter-event times following the Mittag-Leffler distribution. This process is very useful in several fields of applied and theoretical physics including models for anomalous diffusion. Contrary to the well-known Poisson process, the fractional Poisson process does not have stationary and independent increments. It is not a Lévy process and it is not a Markov process. In this letter, we present formulae for its finite-dimensional distribution functions, fully characterizing the process. These exact analytical results are compared to Monte Carlo simulations.

From a loose mathematical point of view, counting processes $N(A)$ are stochastic processes that count the random number of points in a set A . They are used in many fields of physics and other applied sciences. In this letter, we will consider one-dimensional real sets with the physical meaning of time intervals. The points will be incoming events whose duration is much smaller than the inter-event or inter-arrival waiting time. For instance, counts from a Geiger-Müller counter can be described in this way. The number of counts, $N(\Delta t)$, in a given time interval Δt is known to follow the Poisson distribution

$$\mathbb{P}(N(\Delta t) = n) = \exp(-\lambda\Delta t) \frac{(\lambda\Delta t)^n}{n!}, \quad (1)$$

where λ is the constant rate of arrival of ionizing particles. Together with the assumption of independent and stationary increments, Eq. (1) is sufficient to define the *homogeneous* Poisson process. Curiously, one of the first occurrences of this process in the scientific literature was connected to the number of casualties by horse kicks in the Prussian army cavalry [1]. The Poisson process is strictly related to the exponential distribution. The inter-arrival times τ_i identically follow the exponential distribution and are independent random variables. This means that the Poisson process is a prototypical *renewal* process. A justification for the ubiquity of the Poisson process has to do with its relationship with the binomial distribution. Suppose that the time interval of interest $(t, t + \Delta t)$ is di-

vided into n equally spaced sub-intervals. Further assume that a counting event appears in such a sub-interval with probability p and does not appear with probability $1 - p$. Then, $\mathbb{P}(N(\Delta t) = k) = \text{Bin}(k; p, n)$ is a binomial distribution of parameters p and n and the expected number of events in the time interval is given by $\mathbb{E}[N(\Delta t)] = np$. If this expected number is kept constant for $n \rightarrow \infty$, the binomial distribution converges to the Poisson distribution of parameter $\lambda = \mathbb{E}[N(\Delta t)]/\Delta t$, while, in the meantime, $p \rightarrow 0$. However, it can be shown that many counting processes with non-stationary increments converge to the Poisson process after a transient period. It is sufficient to require that they are renewal process (i.e. they have independent and identically distributed (iid) inter-arrival times) and that $\mathbb{E}(\tau_i) < \infty$. In other words, many counting processes with non-independent and non-stationary increments behave as the Poisson process if observed long after the transient period.

In recent times, it has been shown that heavy-tailed distributed inter-arrival times (for which $\mathbb{E}(\tau_i) = \infty$) do play a role in many phenomena such as blinking nano-dots [2, 3], human dynamics [4, 5] and the related inter-trade times in financial markets [6, 7].

Among the counting processes with non-stationary increments, the so-called *fractional Poisson process* [8], $N_\beta(t)$, is particularly important because it is the thinning limit of counting processes related to renewal pro-

cesses with power-law distributed inter-arrival times [9, 10]. Moreover, it can be used to approximate anomalous diffusion ruled by space-time fractional diffusion equations [9, 11–16]. It is a straightforward generalization of the Poisson process defined as follows. Let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed positive random variables with the meaning of inter-arrival times and let their common cumulative distribution function (cdf) be

$$F_{\tau}(t) = \mathbb{P}(\tau \leq t) = 1 - E_{\beta}(-t^{\beta}), \quad (2)$$

where $E_{\beta}(-t^{\beta})$ is the one-parameter Mittag-Leffler function, $E_{\beta}(z)$, defined in the complex plane as

$$E_{\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\beta + 1)} \quad (3)$$

evaluated in the point $z = -t^{\beta}$ and with the prescription $0 < \beta \leq 1$. In equation (3), $\Gamma(\cdot)$ is Euler's Gamma function. The sequence of the *epochs*, $\{T_n\}_{n=1}^{\infty}$, is given by the sums of the inter-arrival times

$$T_n = \sum_{i=1}^n \tau_i. \quad (4)$$

The epochs represent the times in which events arrive or occur. Let $f_{\tau}(t) = dF_{\tau}(t)/dt$ denote the probability density function (pdf) of the inter-arrival times, then the probability density function of the n -th epoch is simply given by the n -fold convolution of $f_{\tau}(t)$, written as $f_{\tau}^{*n}(t)$. In Ref. [10], it is shown that

$$f_{T_n}(t) = f_{\tau}^{*n}(t) = \beta \frac{t^{n\beta-1}}{(n-1)!} E_{\beta}^{(n)}(-t^{\beta}), \quad (5)$$

where $E_{\beta}^{(n)}(-t^{\beta})$ is the n -th derivative of $E_{\beta}(z)$ evaluated in $z = -t^{\beta}$. The counting process $N_{\beta}(t)$ counts the number of epochs (events) up to time t , assuming that $T_0 = 0$ is an epoch as well, or, in other words, that the process begins from a renewal point. This assumption will be used all over this paper. $N_{\beta}(t)$ is given by

$$N_{\beta}(t) = \max\{n : T_n \leq t\}. \quad (6)$$

In Ref. [9], the fractional Poisson distribution is derived and it is given by

$$\mathbb{P}(N_{\beta}(t) = n) = \frac{t^{\beta n}}{n!} E_{\beta}^{(n)}(-t^{\beta}). \quad (7)$$

Eq. (7) coincides with the Poisson distribution of parameter $\lambda = 1$ for $\beta = 1$. In principle, equations (3) and (7) can be directly used to derive the fractional Poisson distribution, but convergence of the series is slow. Fortunately, in a recent paper, Beghin and Orsingher proved that

$$E_{\beta}^{(n)}(-t^{\beta}) = \frac{n!}{t^{\beta n}} \int_0^{\infty} F_{S_{\beta}}(t; u) \left[\frac{\exp(-u)u^{n-1}}{(n-1)!} - \frac{\exp(-u)u^n}{n!} \right] du, \quad (8)$$

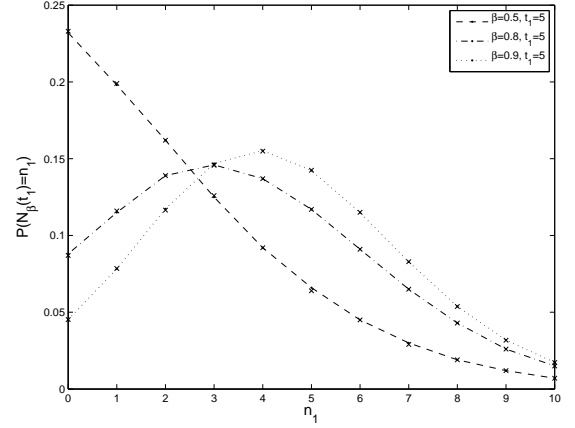


Figure 1: $P(N_{\beta}(T_1) = n_1)$ as function of n_1 for three different values of β . The crosses are estimations obtained from 10^5 Monte Carlo samples and the lines are given to guide the eye.

where $F_{S_{\beta}}(t; u)$ is the cdf of a stable random variable $S_{\beta}(\nu, \gamma, \delta)$ with index β , skewness parameter $\nu = 1$, scale parameter $\gamma = (u \cos \pi\beta/2)^{1/\beta}$ and location $\delta = 0$ [17]. The integral in equation (8) can be evaluated numerically and Fig. 11 shows $\mathbb{P}(N_{\beta}(t) = n)$ for three different values of β . The Monte Carlo simulation of the fractional Poisson process is based on the algorithm presented in equation (20) of Ref. [14].

As a consequence of Kolmogorov's extension theorem, in order to fully characterize the stochastic process $N_{\beta}(t)$, one has to derive its finite dimensional distributions. A further requirement on the process' paths uniquely determines the process, namely that they are right-continuous step functions with left limits [18]. The finite-dimensional distributions are the multivariate probability distribution functions $\mathbb{P}(N_{\beta}(t_1) = n_1, N_{\beta}(t_2) = n_2, \dots, N_{\beta}(t_k) = n_k)$ with $t_1 < t_2 < \dots < t_k$ and $n_1 \leq n_2 \leq \dots \leq n_k$. We have already given the formula for the one-point functions in Eq. (7). The general finite dimensional distribution can be computed observing that the event $\{N_{\beta}(t_1) = n_1, N_{\beta}(t_2) = n_2, \dots, N_{\beta}(t_k) = n_k\}$ is equivalent to $\{0 < T_{n_1} < t_1, T_{n_1+1} > t_1, t_1 < T_{n_2} < t_2, T_{n_2+1} >$

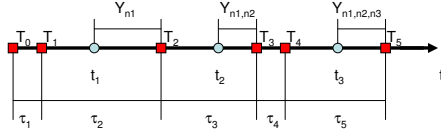


Figure 2: (Color online) Pictorial illustration of the random variables used in the text. The light blue dots represent the observation points t_1, t_2 and t_3 . The red squares are the epochs $T_0 = 0, T_1, \dots, T_5$. The conditional residual life-time is the time elapsed between t_i and the next epoch T_{n_i+1} . It depends on previous values of n_i , this is the number of events between 0 and t_i , with the event at $t = T_0 = 0$ not considered. Here, we have $n_1 = 1, n_2 = 2$ and $n_3 = 4$. All the equations in this paper can be derived by analyzing this figure.

$t_2, \dots, t_{k-1} < T_{n_k} < t_k, T_{n_k+1} > t_k\}$. Therefore, we find

$$\begin{aligned} & \mathbb{P}(N_\beta(t_1) = n_1, N_\beta(t_2) = n_2, \dots, N_\beta(t_k) = n_k) = \\ & \mathbb{P}(0 < T_{n_1} < t_1, T_{n_1+1} > t_1, t_1 < T_{n_2} < t_2, T_{n_2+1} > t_2, \\ & \dots, t_{k-1} < T_{n_k} < t_k, T_{n_k+1} > t_k) = \\ & \int_0^{t_1} du_1 f_\tau^{*n_1}(u_1) \int_{t_1-u_1}^\infty du_2 f_\tau(u_2) \\ & \int_{t_1-u_1-u_2}^{t_2-u_1-u_2} du_3 f_\tau^{*(n_2-n_1-1)}(u_3) \int_{t_2-u_1-u_2-u_3}^\infty du_4 f_\tau(u_4) \\ & \dots \int_{t_{k-1}-\sum_{i=1}^{2k-2} u_i}^{t_k-\sum_{i=1}^{2k-2} u_i} du_{2k-1} f_\tau^{*(n_k-n_{k-1}-1)}(u_{2k-1}) \\ & \left[1 - F_\tau \left(t_k - \sum_{i=1}^{2k-1} u_i \right) \right]. \quad (9) \end{aligned}$$

For instance, the two point function is given by

$$\begin{aligned} & \mathbb{P}(N_\beta(t_1) = n_1, N_\beta(t_2) = n_2) = \\ & \mathbb{P}(0 < T_{n_1} < t_1, T_{n_1+1} > t_1, t_1 < T_{n_2} < t_2, T_{n_2+1} > t_2) = \\ & \int_0^{t_1} du_1 f_\tau^{*n_1}(u_1) \int_{t_1-u_1}^\infty du_2 f_\tau(u_2) \\ & \int_{t_1-u_1-u_2}^{t_2-u_1-u_2} du_3 f_\tau^{*(n_2-n_1-1)}(u_3) \\ & [1 - F_\tau(t_2 - u_1 - u_2 - u_3)]. \quad (10) \end{aligned}$$

Let us focus on the two-point case for the sake of illustration. As $N_\beta(t)$ is a counting process, one has

$\mathbb{P}(N_\beta(t_1) = n_1, N_\beta(t_2) = n_2) = \mathbb{P}(N_\beta(t_1) = n_1, N_\beta(t_2) - N_\beta(t_1) = n_2 - n_1)$ and, as a consequence of the definition of conditional probability

$$\begin{aligned} & \mathbb{P}(N_\beta(t_1) = n_1, N_\beta(t_2) - N_\beta(t_1) = n_2 - n_1) = \\ & \mathbb{P}(N_\beta(t_2) - N_\beta(t_1) = n_2 - n_1 | N_\beta(t_1) = n_1) \times \\ & \times \mathbb{P}(N_\beta(t_1) = n_1). \quad (11) \end{aligned}$$

For $\beta = 1$, when the fractional Poisson process coincides with the standard Poisson process, the increments are iid random variables and one has

$$\begin{aligned} & \mathbb{P}(N_1(t_2) - N_1(t_1) = n_2 - n_1 | N_1(t_1) = n_1) = \\ & \mathbb{P}(N_1(t_2) - N_1(t_1) = n_2 - n_1) = \\ & \exp(-(t_2 - t_1)) \frac{(t_2 - t_1)^{(n_2 - n_1)}}{(n_2 - n_1)!}. \quad (12) \end{aligned}$$

On the contrary, for $0 < \beta < 1$, the increment $N_\beta(t_2) - N_\beta(t_1)$ and $N_\beta(t_1)$ are not independent. Note that $N_\beta(t_1)$ can be seen as an increment as $N_\beta(0) = 0$ by definition. However from Eq. (11), the conditional probability of having $n_2 - n_1$ epochs in the interval (t_1, t_2) conditional on the observation of n_1 epochs in the interval $(0, t_1)$ can be written as a ratio of two finite dimensional distribution:

$$\begin{aligned} & \mathbb{P}(N_\beta(t_2) - N_\beta(t_1) = n_2 - n_1 | N_\beta(t_1) = n_1) = \\ & \frac{\mathbb{P}(N_\beta(t_1) = n_1, N_\beta(t_2) = n_2)}{\mathbb{P}(N_\beta(t_1) = n_1)}. \quad (13) \end{aligned}$$

This probability can be evaluated by means of an alternative method, more appealing for a direct and practical understanding of the dependence structure. Let

$$Y_{n_1} \stackrel{\text{def}}{=} [T_{n_1+1} - t_1 | N_\beta(t_1) = n_1] \quad (14)$$

denote the residual lifetime at time t_1 (that is the time to the next epoch or renewal) conditional on $N_\beta(t_1) = n_1$. With reference to Fig. 2, one can see that the conditional probability $\mathbb{P}(N_\beta(t_2) - N_\beta(t_1) = n_2 - n_1 | N_\beta(t_1) = n_1)$ is given by the following convolution integral for $n_2 - n_1 \geq 1$

$$\begin{aligned} & \mathbb{P}(N_\beta(t_2) - N_\beta(t_1) = n_2 - n_1 | N_\beta(t_1) = n_1) = \\ & \int_0^{t_2-t_1} \mathbb{P}(N_\beta(t_2 - t_1 - y) = n_2 - n_1 - 1) f_{Y_{n_1}}(y) dy, \quad (15) \end{aligned}$$

where $f_{Y_{n_1}}(t)$ is the pdf of Y_{n_1} . In the case $n_2 - n_1 = 0$, one has

$$\mathbb{P}(N_\beta(t_2) - N_\beta(t_1) = 0 | N_\beta(t_1) = n_1) = 1 - F_{Y_{n_1}}(t_2 - t_1) \quad (16)$$

where $F_{Y_{n_1}}(y)$ is the cdf of Y_{n_1} . The distribution of the conditional residual lifetime Y_{n_1} can be evaluated in several ways. For instance, one can notice that it can be decomposed as follows

$$Y_{n_1} = \tilde{\tau}_{n_1+1} + U_{n_1} \quad (17)$$

where U_{n_1} is defined as

$$U_{n_1} \stackrel{\text{def}}{=} [T_{n_1} | N_\beta(t_1) = n_1], \quad (18)$$

and is the position of the last epoch before t_1 conditional on $N_\beta(t_1) = n_1$, and

$$\tilde{\tau}_{n_1+1} \stackrel{\text{def}}{=} [\tau_{n_1+1} - t_1 | T_{n_1+1} > t_1] \quad (19)$$

is the difference between τ_{n_1+1} and t_1 conditional on $T_{n_1+1} > t_1$. The pdf of U_{n_1} is given by the following chain of equalities

$$\begin{aligned} f_{U_{n_1}}(t) dt &= \mathbb{P}(T_{n_1} \in dt | N_\beta(t_1) = n_1) \\ &= \mathbb{P}(T_{n_1} \in dt | T_{n_1} < t_1, T_{n_1} + \tau_{n_1+1} > t_1) \\ &= \mathbb{P}(T_{n_1} \in dt | T_{n_1} < t_1, \tau_{n_1+1} > t_1 - T_{n_1}) \\ &\stackrel{*}{=} \frac{\mathbb{P}(T_{n_1} \in dt) \int_{t_1-t}^{\infty} \mathbb{P}(\tau_{n_1+1} \in dw)}{\mathbb{P}(T_{n_1} < t_1, \tau_{n_1+1} > t_1 - T_{n_1})} \\ &\stackrel{*}{=} \frac{f_\tau^{*n_1}(t) [1 - F_\tau(t_1 - t)] dt}{\int_0^{t_1} du f_\tau^{*n_1}(u) [1 - F_\tau(t_1 - u)]}, \end{aligned} \quad (20)$$

where we used the independence between T_{n_1} and τ_{n_1+1} ($*$) and $f_{T_{n_1}}(x) = f_\tau^{*n_1}(x)$ ($*$). The pdf of $\tilde{\tau}_{n_1+1}$ is

$$\begin{aligned} f_{\tilde{\tau}_{n_1+1}}(t | U_{n_1}) dt &= \mathbb{P}(\tau_{n_1+1} - t_1 \in dt | T_{n_1+1} > t_1) \\ &= \frac{\mathbb{P}(\tau_{n_1+1} \in dt + t_1)}{\mathbb{P}(\tau_{n_1+1} > t_1 - U_{n_1})} \\ &= \frac{f_\tau(t + t_1) dt}{1 - F_\tau(t_1 - U_{n_1})}. \end{aligned} \quad (21)$$

From Eq. (17), one can write that

$$f_{Y_{n_1}}(t) = \int_0^{t_1} f_{\tilde{\tau}_{n_1+1}}(t - u | u) f_{U_{n_1}}(u) du \quad (22)$$

and this equation leads to

$$f_{Y_{n_1}}(t) = \frac{\int_0^{t_1} du f_\tau^{*n_1}(u) f_\tau(t + t_1 - u)}{\int_0^{t_1} du f_\tau^{*n_1}(u) [1 - F_\tau(t_1 - u)]} \quad (23)$$

that, together with Eq. (7), gives us the probability of the conditional increments in Eq. (15). Notice that, for $n_1 = 0$, one has $f_\tau^{*0}(u) = \delta(u)$ and Eq. (23) reduces to the familiar equation for the residual life-time pdf in the absence of previous renewals

$$f_{Y_0}(t) = \frac{f_\tau(t + t_1)}{1 - F_\tau(t_1)}. \quad (24)$$

This method can be applied to the general multidimensional case. As in Eq. (11) we can write

$$\begin{aligned} \mathbb{P}(N_\beta(t_1) = n_1, \dots, N_\beta(t_k) = n_k, N_\beta(t_{k+1}) = n_{k+1}) &= \\ \mathbb{P}(N_\beta(t_{k+1}) - N_\beta(t_k) = n_{k+1} - n_k | \\ N_\beta(t_1) = n_1, \dots, N_\beta(t_k) = n_k) \times \\ \times \mathbb{P}(N_\beta(t_1) = n_1, \dots, N_\beta(t_k) = n_k) \end{aligned} \quad (25)$$

and the predictive probabilities can be evaluated as

$$\begin{aligned} \mathbb{P}(N_\beta(t_{k+1}) - N_\beta(t_k) = n_{k+1} - n_k | \dots \\ | N_\beta(t_1) = n_1, \dots, N_\beta(t_k) = n_k) = \\ \int_0^{t_{k+1}-t_k} \mathbb{P}(N_\beta(t_{k+1} - t_k - y) = n_{k+1} - n_k - 1) \times \\ \times f_{Y_{n_1, \dots, n_k}}(y) dy, \end{aligned} \quad (26)$$

where we defined

$$Y_{n_1, \dots, n_k} \stackrel{\text{def}}{=} [T_{n_{k+1}} - t_k | N_\beta(t_1) = n_1, \dots, N_\beta(t_k) = n_k]. \quad (27)$$

Again, we can use a decomposition of Y_{n_1, \dots, n_k}

$$Y_{n_1, \dots, n_k} = \tilde{\tau}_{n_k+1} + U_{n_k}, \quad (28)$$

where

$$U_{n_k} \stackrel{\text{def}}{=} [T_{n_k} | N_\beta(t_1) = n_1, \dots, N_\beta(t_k) = n_k], \quad (29)$$

and

$$\tilde{\tau}_{n_k+1} \stackrel{\text{def}}{=} [\tau_{n_k+1} - t_k | T_{n_k+1} > t_k]. \quad (30)$$

The difference with the two-point case is that $U_{n_1} = [T_{n_1} | N_\beta(t_1) = n_1] = [\sum_{i=1}^{n_1} \tau_i | N_\beta(t_1) = n_1]$ must be replaced by

$$U_{n_k} = t_{k-1} + Y_{n_1, \dots, n_{k-1}} + \left[\sum_{i=n_{k-1}+1}^{n_k} \tau_i | N_\beta(t_k) = n_k \right]. \quad (31)$$

The time between t_{k-1} and the next renewal epoch is $Y_{n_1, \dots, n_{k-1}}$ and it is independent from $\sum_{i=n_{k-1}+1}^{n_k} \tau_i$. Therefore, the convolution

$$q(n_1, \dots, n_k; t) = f_{Y_{n_1, \dots, n_{k-1}}} * f_\tau^{*(n_k - n_{k-1} - 1)}(t) \quad (32)$$

replaces $f_\tau^{*n_1}(t)$ in Eq. (20). This leads to

$$\begin{aligned} f_{U_{n_k}}(z) = \\ \frac{q(n_1, \dots, n_k; t + t_{k-1}) [1 - F_\tau(t_k - t)]}{\int_{t_{k-1}}^{t_k} q(n_1, \dots, n_k; u + t_{k-1}) [1 - F_\tau(t_k - u)] du}. \end{aligned} \quad (33)$$

On the other hand, $f_{\tilde{\tau}_{n_k+1}}(t)$ has the same functional form as $f_{\tilde{\tau}_{n_1+1}}(t)$ given in Eq. (21) with U_{n_k} replacing U_{n_1} . Therefore, Y_{n_1, \dots, n_k} has the following pdf

$$\begin{aligned} f_{Y_{n_1, \dots, n_k}}(t) = \\ \frac{\int_{t_{k-1}}^{t_k} du q(n_1, \dots, n_k; u + t_{k-1}) f_\tau(t + t_k - u)}{\int_{t_{k-1}}^{t_k} du q(n_1, \dots, n_k; u + t_{k-1}) [1 - F_\tau(t_k - u)]}. \end{aligned} \quad (34)$$

In practice, the random variable $Y_{n_1, \dots, n_{k-1}}$ carries the memory of the observations made at times t_1, \dots, t_{k-1} ; the knowledge of $f_{Y_{n_1, \dots, n_{k-1}}}$ allows the computation of $f_{Y_{n_1, \dots, n_k}}$, and, via Eqs. (25) and (26), the $k + 1$ -dimensional distribution can be derived as well.

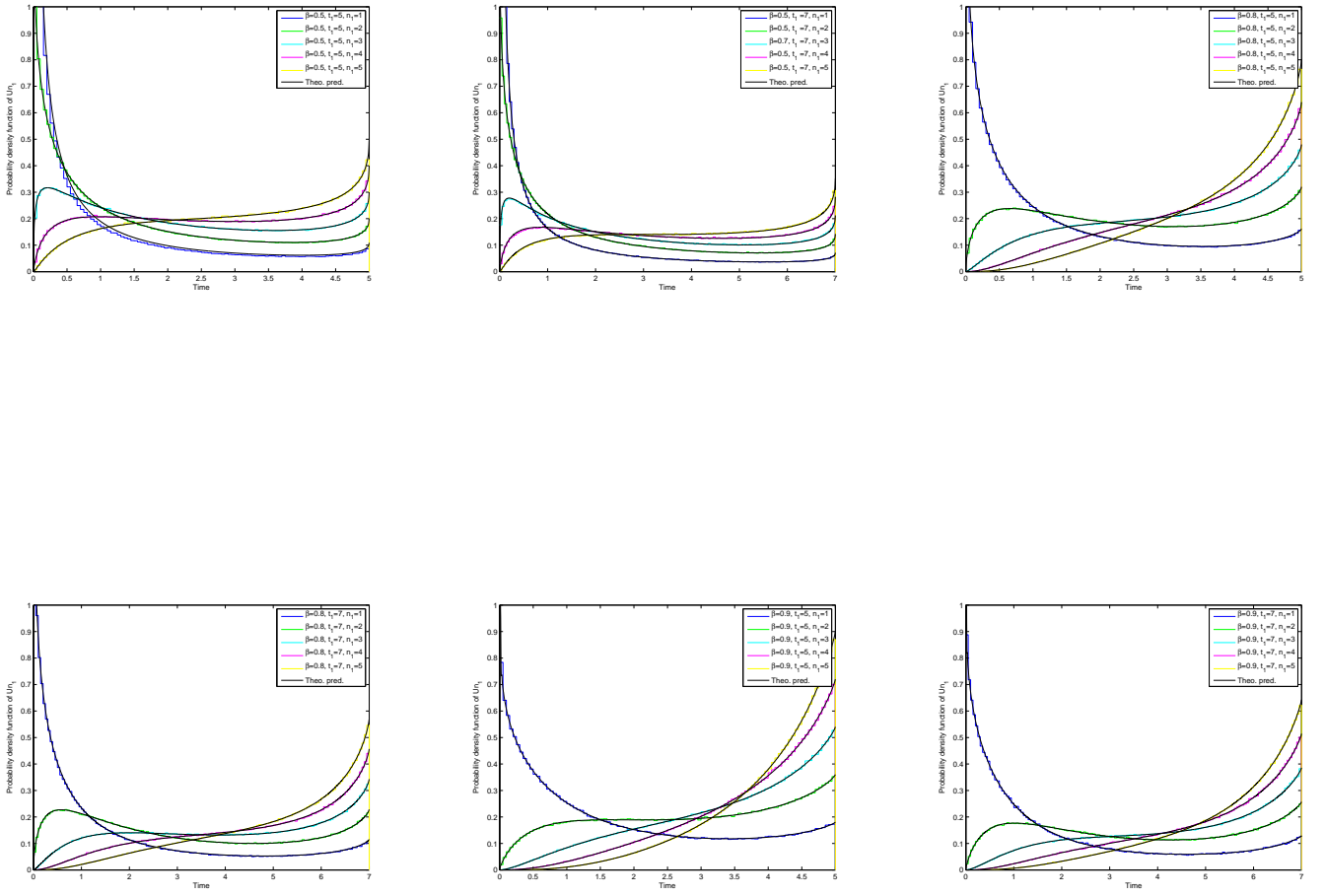


Figure 3: (Color online) Pdf of the random variable U_{n_1} as given in Eq. (20) (solid black lines) compared to Monte Carlo simulations (colored step lines) for three values of β and two different values of t_1 . 10^7 different paths were simulated for each value of β and the bin width is 0.05. Time is in arbitrary units.

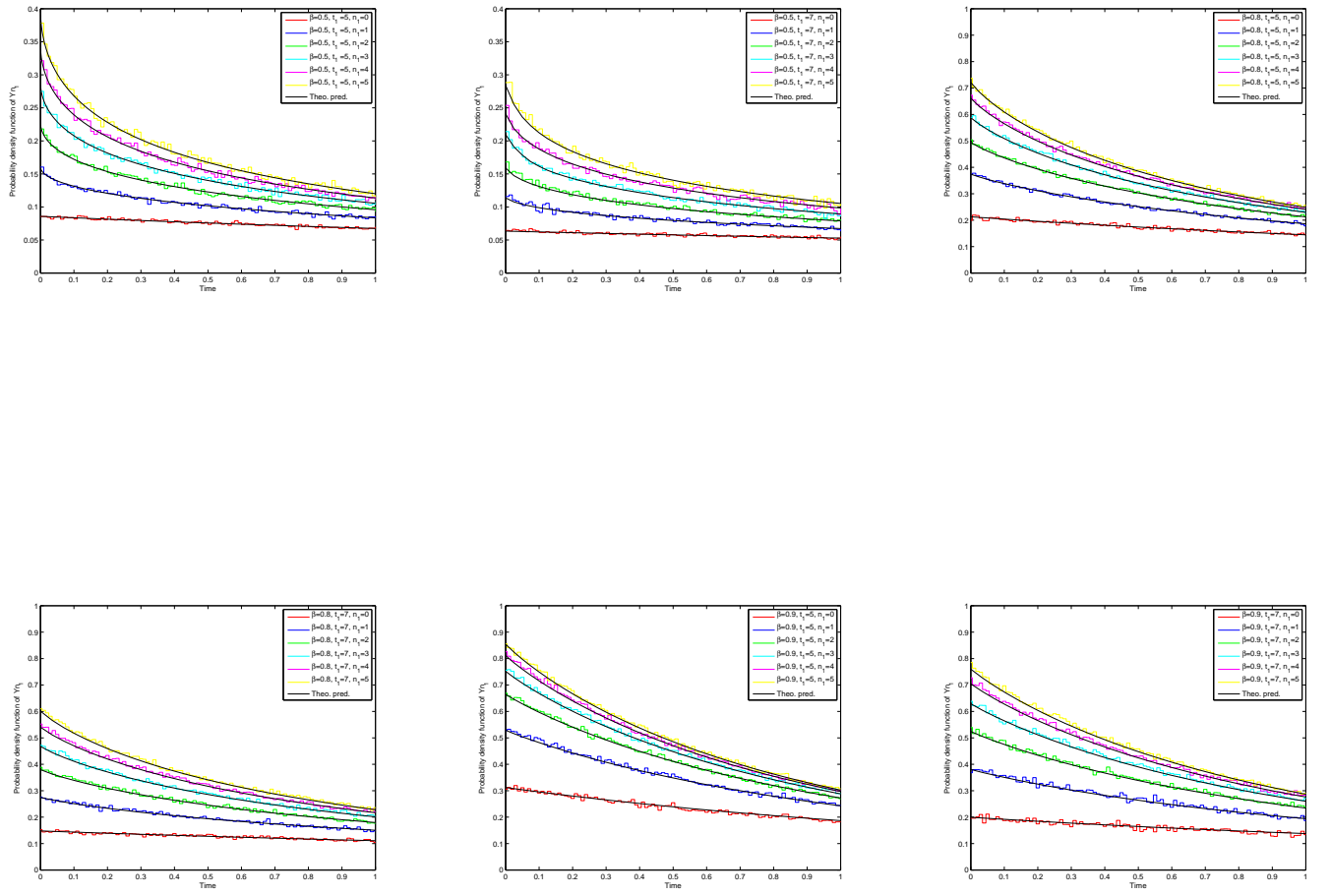


Figure 4: (Color online) Pdf of the random variable Y_{n_1} as given in Eqs. (23) and (24) (solid black lines) compared to Monte Carlo simulations (colored step lines) for three values of β and two different values of t_1 . 10^7 different paths were simulated for each value of β and the bin width is 0.01. Time is in arbitrary units.

Figs. 3 and 4 compare the theoretical results of Eqs. (20), (23) and (24) with those of a Monte Carlo simulation based on the algorithm presented in equation (20) of Ref. [14].

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